Math 279 Lecture 19 Notes

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1 Finishing Hairer's Reconstruction Theorem and Introduction to Regularity Structures

1.1 Finishing the proof of Hairer's reconstruction theorem

We have been proving the following theorem.

Theorem 1.1 (Hairer's reconstruction theorem). *If G is γ-coherent, then there is a distribution T such that*

$$|(T - F_x)(\psi_x^\delta)| \lesssim \begin{cases} \delta^\gamma & \gamma \neq 0 \\ |\log \delta| & \gamma = 0. \end{cases}$$

Proof. Last time, we proved this when $\gamma \leq 0$. The proof we offered last time would yield the following estimate for $\gamma > 0$: Recall that

$$T = \lim_{n \to \infty} T_n, \qquad T_n(y) = F_y(\widehat{\rho}_y^n), \qquad T = T_1 + \sum_{k=1}^\infty (T_{k+1} - T_k),$$

$$F_x = \lim_{n \to \infty} G_n, \qquad G_n(y) = F_x(\widehat{\rho}_y^n), \qquad F_x = G_1 + \sum_{k=1}^\infty (G_{k+1} - G_k).$$

Last time, we proved that

$$\sum_{k : 2^{-k} \leq \delta} |\langle (T_{k+1} - T_k) - (G_{k+1} - G_k), \psi_x \delta \rangle| \lesssim \delta^{-\gamma}, \qquad \gamma > 0.$$

It remains to show

$$|\langle T_n - G_n, \psi_x^\delta \rangle| \lesssim \delta^\gamma,$$

provided that $\delta \approx 2^{-n}$, or more specifically, $2^{-n} \leq \delta < 2^{-n+1}$. Observe that

$$T_n(y) - G_n(y) = (F_y - F_x)(\widehat{\rho}_y^n), \qquad \rho = \varphi * \eta, \quad \eta = \varphi^2.$$

Hence,

$$(T_n - G_n)(y) = \int (F_y - F_x)(\widehat{\varphi}_z^n)\widehat{\eta}_y^n(z)\,dz$$

Now

$$|\langle (T_n - G_n), \psi_x^\delta\rangle| = \iint \underbrace{(F_y - F_x)}_{F_y - F_z + F_z - F_x} (\widehat{\varphi}_z^n)\widehat{\eta}_y^n(z)\psi_x^\delta(y)\,dy\,dz$$

Using the coherence,

$$\begin{aligned}
&\lesssim \iint [\underbrace{2^{n\tau}(|y-z| + 2^{-n})^{\gamma+\tau}}_{2^{-\gamma n}} + \underbrace{2^{n\tau}(|x-z| + 2^{n})^{\gamma+\tau}}_{2^{-\gamma}}]|\widehat{\eta}_y^n(z)\psi_x^\delta(y)|\,dy\,dz \\
&\lesssim 2^{-\gamma n}\|\eta\|_{L^1}\|\psi\|_{L^1}.
\end{aligned}$$

We are done. □

1.2 Remarks about the reconstruction theorem

Remark 1.1. The way we constructed $T = \lim_{n\to\infty} F_x(\widehat{\rho}_n^n) = \mathcal{R}(F)$ is linear in F. Moreover, if we define

$$|||F|||_{K,\varphi} = \sup_{x,y\in K}\sup_{\delta\in(0,1]} \frac{(F_x - F_y)(\varphi_y^\delta)}{\delta^{-\tau}(|x-y| + \delta)^{\gamma+\tau}},$$

where τ and γ depend on the compact set K, then

$$|(T - F_x)(\psi_x^\delta)| \lesssim |||F|||\delta^\gamma$$

uniformly over $\psi \in \mathcal{D}_r$, $x \in K$, $\delta \in (0,1]$, where

$$\mathcal{D}_r = \{\psi \in \mathcal{D} : \|\psi\|_{C^r} \le 1, \operatorname{supp}\psi \subseteq B_1(0)\}.$$

Remark 1.2. As an example, take $f \in \mathcal{C}^\alpha(\mathbb{R}^d)$, $g \in \mathcal{C}^\beta(\mathbb{R}^d)$, $\alpha, \beta \in (0,1)$, and set $F_x = f(x)\nabla g$. Observe that if $g \in \mathcal{C}^\beta$, then $\nabla g \in \mathcal{C}^{\beta-1}(\mathbb{R}^d)$. By $\mathcal{C}^\tau(\mathbb{R}^d)$ with $\tau < 0$, we mean this: First, pick $r = r(\tau)$ to be the smallest positive integer r such that $-\tau < r$ (or $\tau > -r$). Define

$$[T]_{K,\tau} := \sup_{\delta\in(0,1)}\sup_{\varphi\in\mathcal{D}_r} \frac{|T(\varphi_x^\delta)|}{\delta^T},$$

$$\mathcal{C}^\tau_{\text{loc}} := \{T : [T]_{K,\tau} < \infty \text{ for every } K\}.$$

Then $g \in \mathcal{C}^\beta \implies \nabla g \in \mathcal{C}^{\beta-1}$.

Now

$$(F_x - F_y)(\varphi_x^\delta) = (f(x) - f(y))\nabla g(\varphi_x^\delta)$$

$$
\begin{aligned}
&= -(f(x) - f(y))g(\nabla\varphi_x^\delta) \\
&= \delta^{-1}(f(x) - f(y))g((\nabla\varphi)_x^\delta).
\end{aligned}
$$

Since we are dealing with ∇g, we can replace g by $g - g(x)$ (subtracting a constant). Hence, if $|x - y| \leq 1$, then

$$
\begin{aligned}
|(F_x - F_y)(\varphi_x^\delta)| &\lesssim \delta^{-1}[f]|x - y|^\alpha [g]_\beta \delta^\beta \\
&\leq [f]_\alpha [g]_\beta \delta^{-1}(|x - y| + \delta)^{\alpha+\beta} \\
&= [f]_\alpha [g]_\beta \delta^{-1}(|x - y| + \delta)^{\gamma+1},
\end{aligned}
$$

where $\gamma = \alpha + \beta - 1$. Thus, F is $(-1, \gamma)$-coherent.

Use the theorem to assert that there exists some operator $\Gamma(f, g) = \mathcal{R}(F)$ such that

$$
|(\Gamma(f, g) - f(x)\nabla g)(\psi_x^\delta)| \lesssim [f]_\alpha [g]_\beta \delta^\gamma.
$$

Note that since $\mathcal{R}$ is linear in F, Γ is bilinear and continuous in (f, g). In fact, $\Gamma(f, g)$ is unique if $\gamma > 0$. On the other hand, if $f, g \in C^1$, then $T(y) = f(y)\nabla g(y)$ also satisfies the above inequality. By uniqueness, $\Gamma(f, g) = f\nabla g$ for f, g smooth. The same comment does *not* apply to the case of $\gamma \leq 0$.

Remark 1.3. Our result can be extended to Besov spaces $\mathscr{B}_{p,q}^\gamma$. Roughly, in $\mathscr{B}_{p,q}^\gamma$ we replace the uniform norm in x with L^p norm and uniform in $\delta \in (0, 1)$ with $L^q(\frac{1}{\delta}\, d\delta)$.

1.3 Introduction to regularity structures

For our purposes, we often have various terms in our PDE that involve a local description of various different exponents. To do this in a systematic way, we introduce the theory of **regularity structures**. Here is the set-up.

(i) There is a discrete set $A \subseteq \mathbb{R}$ that is bounded below. Roughly, each α in A represents terms that are in $\mathcal{C}^\alpha$ in our PDE. We always assume $0 \in A$.

(ii) For each α, we have a Banach space T_α with norm $\|\cdot\|_\alpha$. For $T_0 = \mathbb{R} = \text{span}(\mathbb{1})$. Fpr $T_0 = \mathbb{R} = \text{span}(\mathbb{2}) = \langle \mathbb{1} \rangle$,.

(iii) We also consider a group G of linear, continuous transformations $\Gamma : T \to T$ where $T = \bigoplus_{\alpha \in A} T_\alpha$. Moreover, we assume $\tau \in T_\alpha$, $\Gamma\tau - \tau \in \bigoplus_{\beta < \alpha} T_\beta$.

(i)-(iii) yields a structure (A, T, G).

We need a model to turn this abstract stuff into real stuff: $(\pi_x, \Gamma_{x,y} : x, y \in \mathbb{R}^d)$. Here, π_x is a bounded, linear map from $T \to \mathcal{D}'$ with each $\Gamma_{x,y} \in G$ satisfying

$$
\pi_x \Gamma_{x,y} \tau = \pi_y \tau.
$$

In short,

$$
\pi_x \Gamma_{x,y} = \pi_y.
$$

Example 1.1 $(e^{2+\gamma}, \gamma \in (0,1))$**.** Let $d = 1$, $A = \{0, 1, 2\}$, $T_0 = \langle \mathbb{1} \rangle$, $T_1 = \langle X \rangle$, $T_2 = \langle X^2 \rangle$. Then $T = \{\tau = c_0 \mathbb{1} + c_1 X + c_2 X^2 : c_0, c_1, c_2 \in \mathbb{R}\}$, and

$$G = \{\Gamma_h : h \in \mathbb{R}\}, \quad \text{where} \Gamma_h \tau = (c_0 \mathbb{1} + (X + h\mathbb{1})^2 + c_2 (X + h\mathbb{1})^2).$$

Then

$$\begin{aligned} \Gamma_h \tau - \tau &= c_1 h \mathbb{1} + 2 c_2 h X + c_2 h^2 \mathbb{1} \\ &= (c_1 h + c_2 h^2) \mathbb{1} + 2 c_2 h X. \end{aligned}$$